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Potts Model, Dirac Propagator, and Conformational Statistics of Semiflexible Polymers

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A new discretized version of the Dirac propagator in d space and one time dimensions is obtained with the help of the $2d$-state, one-dimensional Potts model. The Euclidean version of this propagator describes all conformational properties of semiflexible polymers. It also describes all properties of fully directed self-avoiding walks. The case of semiflexible copolymers composed of a random sequence of fully flexible and semirigid monomer units is also considered. As a by-product, some new results for disordered one-dimensional Ising and Potts models are obtained. In the case of the Potts model the nontrivial extension of the results to higher dimensions is discussed briefly.

KEY WORDS: Ising model; Potts model; directed self-avoiding walks; Dirac propagator; conformational statistics of semiflexible polymers.

1. INTRODUCTION

A simple model of semiflexible polymers which for the fixed polymer length $\hat{N}$ exhibits a rigid-rod to random-coil type of transition has been well known for some time.[1] Only recently[2,3] was it recognized, however, that this model is directly connected with the Euclidean version of Dirac's propagator, so that the rigid-rod to random-coil transition can be associated with the transition from the ultrarelativistic to the nonrelativistic limit of the Dirac propagator.

It is well known that the path integrals are well defined if and only if there is some systematic discretization procedure so that the path integrals can be understood as some limits of ordinary multidimensional integrals. In the case of the Dirac propagator there are many ways to write the corresponding path integrals. It is not my purpose to give here the com-

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plete list of path integrals all of which produce the Dirac propagator. Such a variety of methods to obtain the Dirac propagator is motivated mostly by the desire to better understand its two-dimensional generalization known as superstrings.[4] Here, however, I am interested in use of the Dirac propagator for semiflexible polymers. For this purpose I am going to present a detailed calculation of the Dirac propagator based on a one-dimensional Potts model. These results were briefly announced in my earlier work.[3]

The rest of this paper is organized as follows. In Section 2 the connection between the Dirac equation and the directed self-avoiding random walks is established. In Section 3, I consider the construction of the $(1+1)$-dimensional Dirac propagator with the help of the one-dimensional Ising model. These results are known, and are included here as a warmup exercise for the more complicated $(d+1)$-dimensional version of the Dirac propagator constructed subsequently with the help of the Potts model. Because the main features can be seen already for the $(1+1)$-dimensional case, the end of the section is devoted to various applications of the obtained results to solutions of semiflexible polymers. Section 4 is solely devoted to obtaining the $(d+1)$-dimensional Dirac propagator using the $2d$-state, one-dimensional Potts model. Here some new methods of solution of the Potts model in a magnetic field are presented. The results are further generalized to the case of quenched disorder, in Section 5. Here the case of random copolymers consisting of a mixture of semiflexible and totally flexible units is considered. As a by-product, some new results for the Potts model are obtained. These results are not restricted to the one-dimensional case and allow one to simplify in principle the calculations of thermodynamic properties of the Potts model (including the case of magnetic field) in higher dimensions. This and other topics are briefly described in Section 6.

2. DIRECTED SELF-AVOIDING WALKS AND THE DIRAC EQUATION

It has become rather customary to consider configurational statistics of linear polymers in terms of ordinary random walks[5] placed on some regular d-dimensional lattice. It has been argued that the above random walk models are adequate for sufficiently long chains for which the *trans–gauche* conformational barriers (to be defined below) are rather low. In this case the weakly temperature-dependent length scale l, called the persistence length, can be associated with the effective size of lattice link (the effective monomer size), so that beyond the scale l, polymers are considered as totally flexible. This picture becomes incorrect when the temperature of the

environment is lowered (so that the conformational barriers become increasingly high) or for sufficiently short chains, i.e., those chains which are comparable in length to l.

To model such types of polymers it is useful to consider the modification of the ordinary random walk known as the directed self-avoiding walk (DSAW) model. Although this model was introduced rather long ago by Feynman and Hibbs as Problem 2-6 in their famous book,[6] it was only recently and independently rediscovered by people working in statistical mechanics.[7–9] Such a situation can be explained, perhaps, by the fact that the solution of Problem 2-6 was published much later as a research paper[10] and was left mainly unnoticed. In ref. 10 only the case of $d=2$ was considered in detail, while in ref. 3 the generalization to arbitrary d was given without derivation. Here I provide all the necessary details.

Following Feynman and Hibbs,[6] begin with the consideration of the $(1+1)$-dimensional case [one space (x) and one time (t) dimension] in order to fix the notations. Consider the square lattice with ε being the size of a link. Rotate now the lattice counterclockwise by 45° with respect to some chosen lattice corner, which from now on is considered to be an origin of the coordinate system such that the coordinate axes go through the diagonals of the lattice squares: the x axis is allowed to have both positive and negative values, while the t axis is restricted to have only nonnegative coordinate values. Consider now a motion of a particle on such a lattice. The particle can move straight (*trans*) or change its direction (*gauche*) by 90° at the corners of the lattice. The particle can move back and forth along the x axis, but only forward along the t axis. Let $a_1=(x_1, t_1)$ and $a_2=(x_2, t_2)$ be the initial and the final positions of the particle on the lattice and let $N(R)$ be the total number of walks of N steps long starting at a_1, ending at a_2, and having exactly R turns along their paths. Following Feynman and Hibbs[6] (see also refs. 3 and 10), one introduces the lattice propagator $K_{\alpha\beta}(a_1, a_2)$ given by (in the system of units $\hbar=c=1$)

$$K_{\alpha\beta}(a_1, a_2)=\sum_{R=0}^{\infty} N_{\alpha\beta}(R)(i\varepsilon m)^R \tag{2.1}$$

where α, and β correspond to the initial (final) orientations of the walk: to the "right" (+) or to the "left" (−). Here the "mass" m represents the "bending" factor (i.e., each turn is associated with some bending energy: to go straight costs nothing, but to change the direction costs m each time; in the case of polymers, it can be, in principle, calculated microscopically) and the imaginary i is introduced because initially the above type of walk is considered in Minkowski space (to make a connection with the Dirac

equation) and only when the explicit form of the propagator is found[10] is the Euclidean rotation going to be made.[2,3] The Euclidean version of the above problem evidently corresponds to the problem of the DSAW.[7-9] Usually one makes an average over the initial positions and sums over the final positions of the walk (as it is done in the continuous limit for the Dirac particle[11]). Because of that, the quantity of interest is actually

$$K(a_1, a_2) = \sum_{\alpha,\beta} K_{\alpha\beta}(a_1, a_2)\, p(\alpha) \tag{2.2}$$

where $p(\alpha)$ can be interpreted as the discrete version of the polarization matrix.[11] Without loss of generality, in the subsequent discussion I shall consider only nonpolarized cases for which $p(+) = p(-) = \frac{1}{2}$ and these factors are absorbed into $K_{\alpha,\beta}$. The actual proof that the above problem is indeed associated with the Dirac propagator can be found in refs. 10 and 12. It is also helpful to know the close connection between the Poisson type of random processes and Dirac's equation[13,14]: the case of an ordinary random walk is related to the stochastic Wiener process (or Brownian motion) in the same way as the DSAW problem is related to the Poisson-type process. If the diffusion equation is related to the Schrödinger equation via replacement of the time variable t by $\pm it$, the same is true for the relation between the telegraph equation and the Dirac equation.[13] In order to provide the most general $(d+1)$-dimensional statistical mechanics representation of Dirac's propagator, the auxiliary $(1+1)$-dimensional results are of some help and I provide them below solely for reader's convenience, closely following the treatments given in refs. 3 and 10.

3. (1+1)-DIMENSIONAL DIRAC PROPAGATOR FROM 1-DIMENSIONAL ISING MODEL AND CONFORMATIONAL STATISTICS OF SEMIFLEXIBLE POLYMERS

The famous fermionization of the two-dimensional Ising model is well known.[15] It is most helpful in studying the vicinity of the critical point. Here I am going to consider just the opposite: "bosonization" of the Dirac fermion with the help of the one-dimensional Ising model.

If N_+ (N_-) represents the total number of steps to the right (left), then one gets $N = N_+ + N_-$ and $M = N_+ - N_-$. In view of the above definitions, the time interval is given by

$$t_2 - t_1 = \frac{N\varepsilon}{\sqrt{2}} \tag{3.1}$$

while the space interval is given by

$$x_2 - x_1 = \frac{M\varepsilon}{\sqrt{2}} \tag{3.2}$$

Here the factor of $1/\sqrt{2}$ appears because our lattice has been rotated 45° counterclockwise with respect to the original position. Introduce now a system of N Ising spins σ_i, $i = 1 - N$, and associate the ith step with σ_i, so that $\sigma_i = +1$ (-1) will correspond to the move of the particle to the right (left). With such a defined rule for spins, the "magnetization" is given by

$$M = N_+ - N_- = \sum_{i=1}^{N} \sigma_i \tag{3.3}$$

The number of turns R in this formalism is given by

$$R = \frac{1}{2} \sum_{i=1}^{N-1} (1 - \sigma_i \sigma_{i+1}) \tag{3.4}$$

With the help of Eqs. (3.3) and (3.4) one can rewrite Eq. (2.1) in the following form:

$$\begin{aligned} K_{\alpha\beta}(a_1, a_2) &= Z_{\sigma_1 \sigma_N}(N, M; j) \\ &= \sum_{\sigma_2 = \pm 1}' \cdots \sum_{\sigma_{N-1} = \pm 1}' \exp\left[j \sum_{i=1}^{N-1} (\sigma_i \sigma_{i+1} - 1) \right] \end{aligned} \tag{3.5}$$

where $j = -(1/2) \ln(i\varepsilon m)$ and the primes on the summation signs indicate that the partition function Z should be actually evaluated under the constraint given by Eq. (3.3). To account for this constraint explicitly, it is convenient to introduce the "momentum" representation given by

$$\begin{aligned} Z_{\sigma_1 \sigma_N}(N, \mu; j) &= \sum_{M=-N}^{N} e^{\mu N} Z_{\sigma_1 \sigma_N}(N, M; j) \\ &= \sum_{\{\sigma\}} \exp\left[\mu \sum_{i=1}^{N} \sigma_i + j \sum_{i=1}^{N-1} (\sigma_i \sigma_{i+1} - 1) \right] \end{aligned} \tag{3.6}$$

where $\{\sigma\}$ represents the spin summations from σ_2 to σ_{N-1} as before. Let $t = t_2 - t_1$ and $x = x_2 - x_1$; then $Z(N, M; j) \to K(t, x)$ and, accordingly, $Z(N, \mu; j) \to K(t, p)$, where the momentum variable p is determined by the relation $M\mu = -ipx$ or

$$\mu = -ip \frac{\varepsilon}{\sqrt{2}} \tag{3.7}$$

Use of the transfer matrix methods now produces ($N \to \infty$)

$$Z_{\sigma_1 \sigma_N}(N, \mu; j) = \lambda_+^N \varphi_+(\sigma_1)\, \varphi_+(\sigma_N) + \lambda_-^N \varphi_-(\sigma_1)\, \varphi_-(\sigma_N) \tag{3.8}$$

where $\lambda_\pm$ and $\varphi_\pm$ are eigenvalues and eigenfunctions of the matrix $H(\sigma_i, \sigma_{i+1})$ given by

$$H(\sigma_i, \sigma_{i+1}) = \exp\left[\frac{\mu}{2}(\sigma_i + \sigma_{i+1}) + j(\sigma_i \sigma_{i+1} - 1)\right] \tag{3.9}$$

In actual computations one has to remember that $\exp(-2j) = i\varepsilon m$ and $\mu = -ip(\varepsilon/\sqrt{2})$. This then produces the matrix [by keeping only the terms of order not higher than $o(\varepsilon)$]

$$\|H\| = \begin{pmatrix} 1 - ip(\varepsilon/\sqrt{2}) & i\varepsilon m \\ i\varepsilon m & 1 + ip(\varepsilon/\sqrt{2}) \end{pmatrix} \tag{3.10}$$

which has eigenvalues

$$\lambda_\pm = 1 \mp i\varepsilon m E \tag{3.11}$$

where $E^2 = 1 + (p^2/2m^2)$. The eigenfunctions accordingly are given by

$$\varphi_+(\sigma = +1) = \left[\frac{1}{2}\left(1 - \frac{p}{m'E}\right)\right]^{1/2} \tag{3.12}$$

$$\varphi_+(\sigma = -1) = -\left[\frac{1}{2}\left(1 + \frac{p}{m'E}\right)\right]^{1/2} \tag{3.13}$$

$$\varphi_-(\sigma = +1) = \left[\frac{1}{2}\left(1 + \frac{p}{m'E}\right)\right]^{1/2} \tag{3.14}$$

$$\varphi_-(\sigma = -1) = \left[\frac{1}{2}\left(1 - \frac{p}{m'E}\right)\right]^{1/2} \tag{3.15}$$

where $m' = m\sqrt{2}$.

Taking into account Eqs. (2.2), (3.8), (3.11), and (3.12)–(3.15), one easily obtains the final form of the propagator[10]

$$G(p, t) = \sum_{\alpha, \beta} K_{\alpha\beta}(p, t) = \left(1 - \frac{1}{E}\right) e^{-im'Et} + \left(1 + \frac{1}{E}\right) e^{im'Et} \tag{3.16}$$

In arriving at the last result, Eq. (3.1) has been taken into account and the fact that

$$e = \lim_{\varepsilon \to 0^+} (1 + \varepsilon)^{1/\varepsilon}$$

The proof that Eq. (3.16) does indeed represent Dirac's propagator can be found in ref. 10 and is not going to be reproduced here. An alternative proof can be found in ref. 12. If the multidimensional situation is being considered, then one should formally replace p^2 by $\mathbf{p}^2$ in the above equation, as can be easily seen from the consideration of the nonrelativistic limit $(p^2/m^2 \ll 1)$ of the propagator, Eq. (3.16). In this limit $E \approx 1 + (p^2/2m'^2)$ and one obtains immediately

$$G(\mathbf{p}, t) \simeq 2e^{im't} e^{i(\mathbf{p}^2/2m')t} \tag{3.17}$$

which coincides with the result given in Feynman and Hibbs.[6] This concludes the consideration of the $(1+1)$-dimensional case.

In previous work[2,3] I have provided enough evidence that all the conformational properties of semiflexible polymers are described by Dirac's propagator. Here I present a brief summary of the results in order to extend them to higher dimensions (Section 4) and to the case of quenched disorder to be considered in Section 5.

The standard random walk which describes the conformational statistics of fully flexible polymers[5,15] (without the excluded-volume effects) is characterized in terms of the end-to-end Gaussian distribution function $G(\mathbf{R}, N)$, $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$, where $\mathbf{R}_{1(2)}$ is the initial (final) position of the walk of N effective steps with respect to some chosen system of coordinates. The single-chain partition function is $Z/V = \int d\mathbf{R}\, G(\mathbf{R}, N)$, where V is the volume of the system (e.g., polymer plus solvent). With help of $G(\mathbf{R}, N)$ the important moments, e.g., $\langle \mathbf{R}^2 \rangle$, etc., can be obtained which characterize the size of the macromolecule in solution. Evidently,

$$\langle \mathbf{R}^2 \rangle \propto \frac{\partial}{\partial \mathbf{p}^2} \ln G(\mathbf{p}, N) \Big|_{\mathbf{p}=\mathbf{0}} \tag{3.18}$$

In the case of standard Brownian motion, $G(\mathbf{p}^2, N) = \exp(-A\mathbf{p}^2\hat{N})$, with A a known constant. Use of Eq. (3.18) produces then $\langle \mathbf{R}^2 \rangle \alpha \hat{N}$, as expected. For the semiflexible polymers it has been known for quite some time[1] that

$$\langle \mathbf{R}^2 \rangle = 2\hat{N}\eta - 2\eta^2 \left[1 - \exp\left(-\frac{\hat{N}}{\eta} \right) \right] \tag{3.19}$$

where η is the phenomenological rigidity parameter, which is connected with the mass of the Dirac particle.[2,3] To reproduce the above result, it is sufficient to use the Euclidean version of the Dirac propagator, Eq. (3.16). Indeed, performing the usual rotation $\pm it \to \hat{N}$, where the "+" or "−" sign is determined by further convenience and the parameter $\hat{N}$ is related to the

number of effective links in the way described below, one obtains, instead of Eq. (3.16), the following result[2,3]:

$$G(\mathbf{p}, \hat{N}) = 2\cosh(m'E\hat{N}) + \frac{2}{E}\sinh(m'E\hat{N}) \tag{3.20}$$

The combined use of Eqs. (3.18) and (3.20) produces

$$\langle \mathbf{R}^2 \rangle = \frac{1}{G}\frac{\partial G}{\partial E}\frac{\partial E}{\partial \mathbf{p}^2}\bigg|_{\mathbf{p}=0} = \frac{\hat{N}}{2m'} - \frac{1}{4m'^2}(1 - e^{-2m'\hat{N}}) \tag{3.21}$$

If now I choose $m' = (2\sqrt{2}\,\eta)^{-1}$ and $\hat{N} \Rightarrow \hat{N}\sqrt{2}$, Eq. (3.19) is reproduced. The relationship $\hat{N} \Rightarrow \hat{N}\sqrt{2}$ is obvious from the geometry of the lattice, as described in Section 2, while the connection between η and m (or m') essentially establishes a link between the phenomenological (macroscopic) rigidity parameter and the microscopic mass parameter related to the *trans–gauche* bending energy ΔE as $\varepsilon m = (z-2)\exp(-\Delta E/T)$, as was explained in Refs. 2 and 3. Here z is the coordination number of the lattice.

Equation (3.21) can be given a thermodynamic interpretation which is going to be used below. Taking into account Eqs. (2.2) and (3.6) and keeping in mind that $\mu = -ip(\varepsilon/\sqrt{2})$, one obtains

$$\langle \mathbf{R}^2 \rangle = \left(\frac{\sqrt{2}}{i\varepsilon}\right)^2 \chi N \tag{3.22}$$

where the susceptibility χ is defined in the usual way as

$$\chi = \frac{\partial^2}{\partial \mu^2}\left[\frac{1}{N}\ln Z\right]\bigg|_{\mu=0} \tag{3.23}$$

To understand better the nature of the relationship $\varepsilon m = (z-2)\exp(-\Delta E/T)$, following Flory,[16] consider the problem of packing a monodisperse ensemble of semiflexible polymers onto some arbitrary lattice with coordination number z. Concentrating our attention on a particular polymer chain modeled by the semiflexible random walk, one is confronted with the following situation. After a given step is made for the walker there are two options: (a) to keep going in the same direction (*trans*) or (b) to change the direction (*gauche*), provided that the immediate self-reversals are excluded. The first option is weighted with the conditional probability one, while the second is weighted with the factor $(z-2)\exp(-\Delta E/T)$ (where ΔE can be calculated, in principle, quantum mechanically).

According to Flory,[16] the average number of turns R per lattice link is given by

$$R=\frac{(z-2)\exp(-\Delta E/T)}{1+(z-2)\exp(-\Delta E/T)} \tag{3.24}$$

In order to obtain the above result from our formalism, it is sufficient to use a single-chain partition function $Z=G(\mathbf{p}=\mathbf{0}, \hat{N})$, which, if boundary effects are neglected, is given by Eq. (3.6) with $\mu=0$. Introducing dual variables $S_i=\sigma_i\sigma_{i+1}$, one obtains at once the result for Z,

$$\frac{1}{N}\ln Z=\frac{1}{N}\ln Z(N, 0; j)=-j+\ln 2\cosh j \tag{3.25}$$

From here, taking into account Eqs. (3.4) and (3.5), one obtains

$$R=-\frac{1}{2}\frac{\partial}{\partial j}\frac{1}{N}\ln Z=\frac{1}{2}(1-\tanh j) \tag{3.26}$$

Recall now that $j=-\frac{1}{2}\ln(\varepsilon m)$ (Euclidean version). Finally, if one puts $\varepsilon m=(z-2)\exp(-\Delta E/T)$ in the previous equation and substitutes this result into Eq. (3.25), one obtains back Flory's result, Eq. (3.24). Now using Eqs. (3.6) and (3.25), one can obtain as well the following result for Z:

$$\frac{1}{N}\ln Z=\ln\left[1+(z-2)\exp\left(-\frac{\Delta E}{T}\right)\right] \tag{3.27}$$

The last result also coincides with that obtained by Flory.[16] Use of Eq. (3.20) produces as well

$$\frac{1}{N}\ln Z=1+\varepsilon m \tag{3.28}$$

which evidently coincides with Eq. (3.27), taking into account the limiting formula for e and that $N=\hat{N}/\varepsilon$.

Further developments can be found in ref. 3. In arriving at the result (3.21), I have used the $(1+1)$-dimensional version of the Dirac propagator, silently assuming that the replacement $p^2\to\mathbf{p}^2$ in Eq. (3.17) is sufficient in order to extend the above results to higher dimensions. Although by requirements of continuity and homogeneity of space this, evidently, should be the case, it is desirable to provide more direct evidence that this is indeed the case. This is accomplished in the next section.

4. $(d+1)$-DIMENSIONAL DIRAC PROPAGATOR FROM ONE-DIMENSIONAL, $2d$-STATE POTTS MODEL

It is well known that the case of the Ising model can be considered as a special case of the Potts model.[17] Because of this observation, it is possible to generalize the results presented in Section 3 to the multidimensional case by using the Potts model. This was already announced in ref. 3. Here I provide all necessary details related to this case.

By analogy with Eq. (3.4), one can define now the number of turns R as

$$R=\sum_{i=1}^{N-1}(1-\delta_{\lambda_i,\lambda_{i+1}}) \tag{4.1}$$

where the Potts variable λ takes values $\lambda=1, \omega,..., \omega^{2d-1}$, $\omega=\exp\{2\pi i/2d\}$. Equations (3.1) and (3.2) remain the same, while Eq. (3.3) is replaced by

$$M_i=2\sum_{j=1}^{N}(\delta_{\lambda_j,\lambda^{(i)}}-\tfrac{1}{2}) \tag{4.2}$$

with $i=1-d$, $\lambda^{(i)}$ is equal to one of the possible values of λ associated with the ith positive spatial direction of the lattice (rotated by 45° as before), irrespective of the actual position of the jth link. With such defined R and M_i one can now easily write the partition function for a general d-dimensionally, fully-directed walk. It is given by

$$Z_{\lambda_1\lambda_N}(N,\mu_1,...,\mu_d;j)=\sum_{\{\lambda\}}\exp\left[2\sum_{l=1}^{d}\mu_l\sum_{i=1}^{N}(\delta_{\lambda_i,\lambda^{(l)}}-\tfrac{1}{2})\right]\times\exp\left[j\sum_{i=1}^{N-1}(\delta_{\lambda_i,\lambda_{i+1}}-1)\right] \tag{4.3}$$

where the Potts summation (λ) runs over the sites from 2 to $N-1$. The one-dimensional Potts model is known to be exactly soluble[18] (at least in zero magnetic field) for arbitrary number of components d; therefore the problem of fully directed walks is solvable as well.[7] In practice, however, there are some technical problems toward the exact solution of Eq. (4.3). Indeed, according to ref. 18, the problem of calculation of Eq. (4.3) can be accomplished by use of transfer matrix methods. In the case of the Potts model the computation of the determinant of the corresponding transfer matrix is possible (for $\mu_l=0$, $l=1-d$) because the determinant for this case is known as a circulant.[19] When at least one of the μ_l is nonzero, the corresponding determinant is no longer a circulant and the standard methods[19] cannot be applied. Because of this, some new methods have to be developed. They are going to be presented below. But first I would like

to consider the simplest case of zero "magnetic" field in order to compare the result obtained by the new method with that given in Eq. (3.27). In the case when the μ_l are all zero one obtains the following $2d \times 2d$ matrix $\|Q\|$, using Eq. (4.3):

$$\|Q\| = e^{-j} \begin{pmatrix} e^j & 1 & \cdots & 1 \\ 1 & e^j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & e^j \end{pmatrix} \tag{4.4}$$

According to results of ref. 18, one obtains the following eigenvalues of Q:

$$\lambda_1 = 1 + (2d-1)\, e^{-j} \tag{4.5}$$

$$\lambda_i = 1 - e^{-j}, \qquad i = 2 - 2d \tag{4.6}$$

Let, as before, $j = -\ln(\varepsilon m)$ and let, furthermore, $\varepsilon m = \exp(-\Delta E/T)$; then for $N \to \infty$ one obtains

$$\frac{1}{N} \ln Z = \ln \left[1 + (2d-1) \exp\left(-\frac{\Delta E}{T} \right) \right] \tag{4.7}$$

When this result is compared with Eq. (3.27), one recognizes that one has to replace $z-2$ by $2d-1$. For the square lattice $z=4$, but, by virtue of the construction of the DSAW, the *effective* coordination number $\hat{z}$ is actually 3, so that $z-2$ should in fact be replaced by $\hat{z}-2$. With such replacement $\hat{z}-2 = 2d-1$, as expected.

Consider now the case when $\mu_l \neq 0$. One can immediately recognize that the determinant of the transfer matrix is no longer a circulant. Although the most general case of *all* $\mu_l \neq 0$ can be, in principle, discussed, in fact there is no need to do this if one is interested only in computation of $\langle \mathbf{R}^2 \rangle$. Taking into account Eqs. (3.21)–(3.23), it is sufficient to consider the case of only one μ_l which is nonzero. Without loss of generality, one may always choose $\mu_l = \mu_1 \equiv \mu$. In this case the corresponding transfer matrix is given by

$$\mathscr{F}(i, i+1) = \exp\{\mu[(\delta_{\lambda_i, \lambda^{(1)}} - \tfrac{1}{2}) + (\delta_{\lambda_{i+1}, \lambda^{(1)}} - \tfrac{1}{2})] + j(\delta_{\lambda_i, \lambda_{i+1}} - 1)\} \tag{4.8}$$

or, more explicitly,

$$\|\mathscr{F}\| = \begin{pmatrix} e^{\mu} & e^{-j} & e^{-j} & \cdots & e^{-j} \\ e^{-j} & e^{-\mu} & e^{-j-\mu} & \cdots & e^{-j-\mu} \\ e^{-j} & e^{-j-\mu} & e^{-\mu} & \cdots & e^{-j-\mu} \\ \vdots & & & & \\ e^{-j} & e^{-j-\mu} & & & e^{-\mu} \end{pmatrix} \tag{4.9}$$

In order to calculate the eigenvalues of the above matrix, consider first an auxiliary problem of computation of the eigenvalues of the matrix $\|T\|$ of the following type:

$$\|T\| = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & & \ddots & \vdots \\ b & \cdots & \cdots & a \end{pmatrix} \tag{4.10}$$

Evidently, the determinant of the above matrix is circulant and it can be computed with the use of standard methods.[19] Because, however, our aim is computation of the determinant of $\|\mathscr{F}\|$, given by, Eq. (4.9), a different method is going to be developed below. Introduce a vector $\mathbf{x} = (x_1, \ldots, x_q)$ and consider the matrix equation

$$\|\mathbf{T} - \lambda \mathbf{I}\| \mathbf{x} = 0 \tag{4.11}$$

Use of Eqs. (4.10) and (4.11) allows one to write

$$\begin{aligned} (a-\lambda)\, x_1 - b x_1 &= -bS \\ (a-\lambda)\, x_2 - b x_2 &= -bS \\ &\vdots \\ (a-\lambda)\, x_q - b x_q &= -bS \end{aligned} \tag{4.12}$$

where $S = \sum_{i=1}^{q} x_i$. This can be equivalently rewritten as

$$\begin{aligned} (a-\lambda-b)\, x_1 &= -bS \\ &\vdots \\ (a-\lambda-b)\, x_q &= -bS \end{aligned} \tag{4.13}$$

from which it follows that if $a-\lambda-b \neq 0$, then $x_1 = x_2 = \cdots = x_q = x(1)$, whence

$$(a-\lambda-b)\, x(1) = -bqx(1) \tag{4.14}$$

or

$$a - b + bq = \lambda$$

Let $a = 1$, $b = e^{-j}$ [see Eq. (4.4)]; then one obtains

$$\lambda_1 = 1 + (q-1)\, e^{-j} \tag{4.15}$$

in complete agreement with Eq. (4.5). Let $a-\lambda-b=0$, which is equivalent to $\lambda_2=\lambda_3=\cdots=\lambda_q=1-e^{-j}$ [see Eq. (4.6)]; then $\mathbf{x}$ is subjected to two constraints, $\sum_{i=1}^{q} x_i(j)=0$ and $\sum_{i=1}^{q} x_i^2(j)=1$, where $j=2-q$. In the case of λ_1 we also have $\sum_{i=1}^{q} x_i^2(1)=1$, but this time we actually obtain $qx^2(1)=1$, or

$$x(1)=\frac{1}{\sqrt{q}} \tag{4.16}$$

For the case $j=2-q$, in order to satisfy both constraints, it is sufficient to choose

$$x_k(j)=\frac{1}{\sqrt{q}}\exp\left(\frac{i2\pi k}{q}j\right) \tag{4.17}$$

where $k=1-q$, $j=1-(q-1)$. Indeed, it is easy to check[17] that

$$\sum_{k=1}^{q}\exp\left(\frac{i2\pi k}{q}j\right)=0 \tag{4.18}$$

and, in addition, one has to use the orthogonality condition

$$\sum_{k=1}^{q} x_k(i)\,x_k^*(j)=\delta_{ij} \tag{4.19}$$

The last results as well as Eq. (4.16) are in agreement with that given in ref. 18. Consider now, in view of Eq. (4.9), the following matrix:

$$\|T_1\|=\begin{pmatrix} a & b & \cdots\cdots & b \\ b & c & d\ \cdots & d \\ & d & c\ \cdots & d \\ \vdots & & \ddots & \vdots \\ b & d & \cdots\cdots & c \end{pmatrix} \tag{4.20}$$

Such a matrix equation analogous to (4.11) produces the following result:

$$\begin{aligned} (a-\lambda-b)\,x_1 &= -bS \\ (c-\lambda-d)\,x_2+(b-d)\,x_1 &= -dS \\ &\vdots \\ (c-\lambda-d)\,x_q+(b-d)\,x_1 &= -dS \end{aligned} \tag{4.21}$$

From here if $c-\lambda-d\neq 0$, one obtains $x_2=x_3=\cdots=x_q=x$, so that $S=x_1+(q-1)x$. If, however, $c-d-\lambda=0$, then one obtains $\lambda=c-d$ on one

hand and $\lambda = b - a + (b/d)(b-d)$ on the other. From here one concludes that, in the general case, there is no such λ which satisfies $\lambda = c - d$. Let, furthermore, $a - \lambda - b = 0$; then one obtains $S = 0$. Using $\lambda = a - b$ in Eq. (4.21), one concludes that the requirement $S = 0$ can be satisfied if and only if $b = d$ and $c - d = a - b$. But this is indeed the case when $p = 0$, because for this case $a = 1$, $b = i\varepsilon m$, $c = 1$, and $d = i\varepsilon m$, whence one obtains back the result (4.17). Let now $a - b \neq \lambda$; then, going back to Eq. (4.21), one obtains

$$\begin{aligned} (a-\lambda)\, x_1 + b(q-1)x &= 0 \\ bx_1 + [c - \lambda - d(q-2)]x &= 0 \end{aligned} \tag{4.22}$$

Consider first the case $q = 2$. An easy calculation shows that

$$\lambda_{1,2} = \frac{c+a}{2} \pm \left[\left(\frac{c+a}{2} \right)^2 + b^2 - ac \right]^{1/2} \tag{4.23}$$

because $a = e^{\mu} \to 1 - ip\varepsilon/\sqrt{2}$, $b = i\varepsilon m$, $c \approx 1 + ip\varepsilon/\sqrt{2}$, and $d \approx i\varepsilon m$ [see Eq. (3.10)], using Eq. (4.23) and keeping only the terms of $O(\varepsilon)$ produces back rsult (3.11), as required. For general q a little algebra produces

$$\begin{aligned} \lambda_{1,2} = {} & \frac{1}{2}[c + a - d(2-q)] \\ & \pm \left\{ \left(\frac{c + a - d(2-q)}{2} \right)^2 + b^2(q-1) - a[c - d(2-q)] \right\}^{1/2} \end{aligned} \tag{4.24}$$

It is of interest to demonstrate first that the roots λ_1 and λ_2 coincide with results already obtained [i.e., $\lambda_1 = 1 + (q-1)\, e^{-j}$ and $\lambda_2 = 1 - e^{-j}$] in the limit when $p \to 0$. In this case one obtains $a = c = 1$, $b = d = i\varepsilon m$. Substitution of these values back into Eq. (4.24) produces the anticipated result. This indicates that the root λ_2 is $(q-1)$-fold degenerate in this limiting case, as before. Let now $p \neq 0$; then one obtains

$$\lambda_{1,2} = 1 - i\varepsilon m \left(1 - \frac{q}{2} \right) \pm i\varepsilon m (q-1)^{1/2}\, \tilde{\tilde{E}} \tag{4.25}$$

where

$$\tilde{\tilde{E}} = \frac{1}{2m^2(q-1)} \left[p - \sqrt{2}\, m \left(1 - \frac{q}{2} \right) \right]^2 + 1 \tag{4.26}$$

For $q=2$ (Ising model case) one recovers back the result (4.13), as required. It is convenient to make the following redefinitions:

$$m=\frac{m'}{[2(q-1)]^{1/2}} \tag{4.27}$$

and

$$p=p'+\sqrt{2}\,m\left(1-\frac{q}{2}\right) \tag{4.28}$$

With such redefined m and p one obtains (omitting the primes)

$$\lambda_{1,2}=1-\frac{i\varepsilon}{2}\frac{m}{[2(q-1)]^{1/2}}(2-q)\pm i\varepsilon\frac{\tilde{E}}{\sqrt{2}} \tag{4.29}$$

where, as before, $\tilde{E}^2=1+p^2/m^2$ [see Eq. (3.11)].

Consider now the computation of the corresponding eigenvectors. Using Eq. (4.22), one obtains

$$x_1=-\frac{b(q-1)}{a-\lambda}x \tag{4.30}$$

Combining this with the normalization condition (4.19), one obtains

$$x=\pm\left[(q-1)\left(1+\frac{b^2(q-1)}{(a-\lambda)^2}\right)\right]^{-1/2} \tag{4.31}$$

As before, it is of interest first to check what happens in the limit $p\to 0$. In this case $a=1$, $b=i\varepsilon m$, $\lambda_1=1+(q-1)\,i\varepsilon m$, $\lambda_2=\lambda_3=\cdots=\lambda_q=1-i\varepsilon m$, and one obtains

$$x(1)=\pm\frac{1}{\sqrt{q}} \tag{4.32}$$

$$x(2)=\cdots=x(q)=\pm\frac{1}{[q(q-1)]^{1/2}} \tag{4.33}$$

In view of Eq. (4.30), the result (4.32) is in agreement with that previously obtained, Eq. (4.16), while Eq. (4.33) is in disagreement with Eq. (4.17). To understand better what has happened, it is instructive to go back to Eq. (4.21). For $\lambda=\lambda_2$ one obtains $a-b=\lambda$ and, thus, $S=0$. Also in this case $b=d$ and $\lambda=c-d$. Whence, the result (4.17) is recovered. The solution given by Eq. (4.33) is, nevertheless, a legitimate solution as long as $p\neq 0$ and cannot be simply discarded: the presence of finite p [i.e., finite

magnetic field, see Eq. (4.3)] breaks the rotational symmetry of solution (4.17), so that the $p \to 0$ and $p = 0$ solutions of the system of equations (4.21) are *different*, but physically meaningful solutions. This statement is supported further by considering the case $p \neq 0$ (see also Section 5) using different methods. In the meantime for $\lambda = \lambda_1$ one obtains using Eq. (4.31) the following result:

$$x = \pm \frac{1}{[2(q-1)]^{1/2}} \left(1 \pm \frac{p'}{m'} \frac{1}{E}\right)^{1/2} \tag{4.34}$$

$E^2 = 1 + p'^2/m'^2$ and p', m' are given by Eqs. (4.27), (4.28). Combining Eqs. (4.30) and (4.34), one obtains as well

$$x_1 = \mp \frac{1}{\sqrt{2}} \left(1 \mp \frac{p'}{m'} \frac{1}{E}\right)^{1/2} \tag{4.35}$$

In the case when $q = 2$ one obtains back Eqs. (3.12)–(3.15), as anticipated. Using Eqs. (4.29), (4.34), and (4.35), it is rather easy to write down the final form of the propagator. In order to do so, one must recall Eqs. (3.1) and (3.16) and recognize a complete symmetry between different μ_l which comes from an analysis of Eq. (4.3). With this remark one obtains

$$G(\mathbf{p}, t) = \exp\left\{-\frac{im'(2-q)}{2[(q-1)]^{1/2}} t\right\}\left[\left(1 - \frac{1}{E}\right) e^{-im'Et} + \left(1 + \frac{1}{E}\right) e^{im'Et}\right] \tag{4.36}$$

where $E^2 = 1 + \mathbf{p}'^2/m'^2$. For $q = 2$ one recovers the Ising case result, Eq. (3.16), as required. Computation of $\langle \mathbf{R}^2 \rangle$ will produce now the same result, Eq. (3.21), in agreement with the discussion presented in Section 3. This concludes the discussion of the Potts model version of the Dirac propagator.

5. CONFORMATIONAL STATISTICS OF SEMIFLEXIBLE CHAINS IN THE PRESENCE OF QUENCHED DISORDER

Recently there has been a great deal of interest both theoretically[20,21] and experimentally[22–25] in studying of solutions of random semiflexible copolymers. Technologically random semiflexible polymers can be rather easily produced.[22] The above type of macromolecules "have varied levels of internal flexibility, but never enough to permit random coil formation. Their characteristic states are generally described as liquid-crystalline."[22] In a set of remarkable experiments by Stupp and co-workers[23–25] two

types of thermotropic copolymers were synthesized and properties of their solutions were studied. Both types of copolymers contained (on average) *equal* amount of three types of monomers. Two of the monomers A and B were rigid, while the third C was flexible. The first copolymer was *ordered* with repeat unit ...CABCBA... and was found to exhibit a sharp nematic to isotropic transition at 275°C. The second copolymer was disordered, but of *the same* chemical composition *on average*. In contrast to the ordered case, solution of such copolymers exhibited a broad biphasic region over the temperature range 250–400°C. In this temperature range an isotropic fluid phase is observed to coexist with a nematic birefringent phase. Pure nematic and isotropic phases, respectively, were observed at temperatures below and above the biphase region.

Because of these findings, it is of interest to extend the results of previous sections to the case of a random Bernoullian type of disorder. The reader may also be interested in applying the methods presented below to other, more sophisticated models of random copolymers which were studied by the author some time ago.[26] It is convenient to subdivide the subsequent developments into two parts by treating Ising and Potts cases separately.

5.1. Conformational Statistic of Ising-Type Random Copolymers

The randomly diluted one-dimensional Ising model has been studied by many authors.[27–29] Unfortunately, one cannot use directly the above results, for reasons which will become apparent.

Consider first the bond diluted Ising model in zero external field ($\mu = 0$) given by Eq. (3.6). Neglecting boundary effects [see Eq. (3.25) and the definition of S_i given there] one has effectively

$$Z(j) = \sum_{\{S\}} \exp\left[\sum_i J_i(S_i - 1)\right] \tag{5.1}$$

where J_i is a random coupling such that $J_i = j$ with probability f and $J_i = 0$ with probability $1 - f$. Performing the spin summation in Eq. (5.1), one obtains

$$Z(j) = \prod_i (2e^{-J_i} \cosh J_i) \tag{5.2}$$

From here one easily obtains

$$\frac{1}{N} \langle \ln Z(j) \rangle_f = \ln 2 - fj + f \ln \cosh j \tag{5.3}$$

which is in formal agreement with the result given by Wortis.[29] More accurate calculation [see e.g., Eq. (3.27)], which accounts for the boundary effects (open chain vs. closed), produces

$$\frac{1}{N}\langle \ln Z(j)\rangle_f = f\ln(1+e^{-2j}) \tag{5.4}$$

where $\langle\cdots\rangle_f$ denotes the operation of the disorder averaging specified above. Although the result for the free energy, Eq. (5.4), was obtained rather effortlessly, this is not the case when the "magnetic" field is present. The previous authors restricted themselves to the computation of susceptibility, which in our case, because of Eqs. (3.22) and (3.23), leads to the computation of $\langle \mathbf{R}^2\rangle$. Fortunately, that is all that is needed for the present case. Use of results obtained by Wortis[29] produces the following result for the susceptibility:

$$\chi = \frac{1+f\tanh j}{1-f\tanh j} \tag{5.5}$$

which for the case $f=1$ produces formally correct (from the point of view of thermodynamics of Ising model[18]) but an absolutely wrong result for $\langle \mathbf{R}^2\rangle$; see Eqs. (3.6), (3.22), and (3.23). It is *for this reason* that the computations given in refs. 27–29 should be reconsidered.

Begin again with Eq. (3.6). Now it can be rewritten as follows:

$$Z_{\sigma_1\sigma_N}(N,\mu;j) = (\cosh\mu)^N \sum_{\{\sigma\}}{}' \prod_{i}^{\text{bonds}} \left(\frac{1+e^{-2J_i}}{2}\right)(1+z_{i,i+1}\sigma_i\sigma_{i+1}) \times \prod_{j}^{\text{sites}} (1+\bar{z}\sigma_j) \tag{5.6}$$

where $z_{i,i+1}=\tanh J_i$ and $\bar{z}=\tanh\mu$. The spin summation, in the case of Eq. (3.6), includes spins from σ_2 to σ_{N-1}. The prime on the summation sign indicates just this. In view of Eqs. (2.2) and (3.16), matters can be simplified because the final result for the propagator, Eq. (3.16), already includes the summation over the remaining spins σ_1 and σ_N. Thus, henceforth the prime on the spin summation is going to be omitted. Because I am interested here only in the computation of $\langle \mathbf{R}^2\rangle$, further simplifications are possible. In particular, one may write

$$\prod_{j}^{\text{sites}} (1+\bar{z}\sigma_j) = 1+\bar{z}\sum_{i=1}^{N}\sigma_i + \bar{z}^2 \sum_{\substack{i,j=1\\ i<j}}^{N} \sigma_i\sigma_j + \cdots \tag{5.7}$$

where the terms denoted by dots are not going to be used in the computations of $\langle \mathbf{R}^2 \rangle$. Now, if $\mu = 0$, one obtains back the result Eq. (5.4) upon bond averaging, if $\mu \neq 0$; however, the only additional terms which are going to survive the spin summation are those which are obtained by contraction between the expansion given by Eq. (5.7) and that given by

$$\prod_{i}^{\text{bonds}} (1 + z_{i,i+1}\sigma_i\sigma_{i+1}) = 1 + \sum_i z_{i,i+1}\sigma_i\sigma_{i+1} + \sum_{i<j} z_{i,i+1}\sigma_i\sigma_{i+1}z_{j,j+1}\sigma_j\sigma_{j+1} + \cdots \quad (5.8)$$

Multiplying Eqs. (5.7) and (5.8) and performing spin summation, one arrives at the following result:

$$\begin{aligned} &\frac{1}{2^N}\sum_{\{\sigma\}} \text{Eq. (5.7)} \times \text{Eq. (5.8)} \\ &\quad = 1 + (N-1)\bar{z}^2 z + (N-2)\bar{z}^2 z^2 + (N-3)\bar{z}^2 z^3 + \cdots + \bar{z}^2 z^{N-1} \\ &\quad = 1 + \bar{z}^2 z^{N-1}[(N-1)z^{-(N-2)} + (N-2)z^{-(N-3)} + \cdots + 1] \end{aligned} \quad (5.9)$$

Here $z_{i,i+1} \equiv z$. In standard treatments[27-29] the finite-size effects are completely ignored. In this case one would obtain instead of Eq. (5.9) the following result:

$$\frac{1}{2^N}\sum_{\{\sigma\}} \text{Eq. (5.7)} \times \text{Eq. (5.8)} = 1 + N\bar{z}^2 \sum_{m=1}^{\infty} z^m = 1 + N\bar{z}^2 \frac{z}{1-z} \quad (5.10)$$

In the nonrandom case on then obtains

$$\frac{1}{N}\ln z = \ln\cosh\mu + \ln(1 + e^{-2j}) + \frac{\bar{z}^2 z}{1-z} \quad (5.11)$$

From here one obtains

$$\chi = \frac{\partial^2}{\partial\mu^2}\left(\frac{1}{N}\ln z\right)_{\mu=0} = \frac{1+z}{1-z} \quad (5.12)$$

which coincides with Eq. (5.5) for $f = 1$. The finite-size effects taken into account in Eq. (5.9) considerably change the final result. Some manipulations with power series given by Eq. (5.9) produce the following result:

$$\text{Eq. (5.9)} = 1 + \bar{z}^2\left[z^N\left(1 + \frac{z}{(1-z)^2}\right) + \frac{N}{1-z} - \frac{z}{(1-z)^2}\right] - \bar{z}^2 z^N \quad (5.13)$$

Recall now that in the nonrandom case

$$z = \tanh j = \frac{1 - e^{-2j}}{1 + e^{-2j}} = \frac{1 - i\varepsilon m}{1 + i\varepsilon m}$$

Because of this, $1 - z = 2i\varepsilon m$; accordingly,

$$\bar{z} = \tanh \mu = \frac{1 - e^{-2\mu}}{1 + e^{-2\mu}} = \frac{-2i\varepsilon p/\sqrt{2}}{2(-2i\varepsilon p/\sqrt{2})} \approx -i\frac{\varepsilon p}{\sqrt{2}}$$

Also, one has

$$\cosh \mu = \frac{1}{2}(e^{\mu} + e^{-\mu}) \approx 1 + \frac{1}{2}\left(\frac{ip\varepsilon}{\sqrt{2}}\right)^2$$

But the last term is of $O(\varepsilon^2)$, while all our calculations were made with accuracy up to $O(\varepsilon)$. Keeping this in mind and combining Eqs. (5.6) and (5.13), one obtains for the nonrandom case

$$\begin{aligned} \ln Z = {} & N \ln(1 + e^{-2j}) \\ & - \left(\frac{\varepsilon p}{\sqrt{2}}\right)^2 \left[e^{-2imt}\left(1 - \frac{1}{4\varepsilon^2 m^2}\right) + \frac{t}{2i\varepsilon^2 m} + \frac{1}{4\varepsilon^2 m^2} \right] \\ & - \left(\frac{\varepsilon p}{\sqrt{2}}\right)^2 e^{-2imt} \end{aligned} \tag{5.14}$$

Performing rotation to the Euclidean time $it \to \hat{N}$, remembering that $t = N/\varepsilon$, using $m = m'/\sqrt{2}$ and $\hat{N} = N\sqrt{2}$ and taking the limit $\varepsilon \to 0$, one obtains

$$\ln Z = N \ln(1 + e^{-2j}) + \mathbf{p}^2 \left(\frac{\hat{N}}{2m'} - \frac{1}{4m'^2}(1 - e^{-m'\hat{N}}) \right) \tag{5.15}$$

From here one obtains again the result (3.21), by using (3.23), i.e.,

$$\langle \mathbf{R}^2 \rangle = \frac{\partial}{\partial \mathbf{p}^2} \ln Z \Big|_{\mathbf{p}=\mathbf{0}}$$

In order to extend the above results to the case of quenched disorder, several comments are in order. First, so far I have silently assumed that the

Bernoullian distribution for J_i, as it was defined, makes sense for our model. This is not the case, however, which can be understood by analyzing the limiting case $f=0$. Going back to Eq. (3.6) and following all the steps which led to the computation of $\langle \mathbf{R}^2 \rangle$, one can easily recognize that the final result is going to be meaningless. Second, the same can be seen also from analysis of Eq. (3.19): When $\eta \to 0$ one obtains the meaningless result $\langle \mathbf{R}^2 \rangle = 0$. Because of this peculiarity, it is meaningful to consider the case of two random couplings j_1 and j_2 so that j_1 will occur with probability f and j_2 with probability $1-f$. Let, furthermore, for instance, $j_1 \gg j_2$ (this does not actually play any role); then one practically recovers the situation with just one random coupling $j=j_1$. Evidently, one can easily extend the present treatment to the case of more than two couplings.

Going back to Eq. (5.9), one needs now to replace everywhere z with $fz_1+(1-f)z_2$. Because $z \approx 1-2i\varepsilon m$, one obtains

$$fz_2+(1-f)z_2 = 1-2i\varepsilon[fm_1+(1-f)m_2] \equiv 1-2i\varepsilon\bar{m}$$

After this, the rest of the calculation proceeds exactly as in the nonrandom case, so that in the final result, for $\langle \mathbf{R}^2 \rangle$, m' should be replaced by $\bar{m}'$. Unlike the treatment given in refs. 20 and 21, in the present case *no* approximations were made and *no* replica tricks were used.

The results here, combined with those of Section 4, enable one to study rigorosly the nematic–isotropic transition for the case of random semiflexible polymers. This is going to be accomplished in a separate publication. In the meantime, it is of pedagogical interest to obtain similar results for the general case of the $2d$-component, one-dimensional Potts model.

5.2. Conformational Statistics of Potts-Type Random Copolymers

Consideration of the Ising model case has helped to understand the main features related to conformational statistics of random copolymers. Because of this, one is left with the study of the Potts analog of Eq. (5.6) with, perhaps, the end points being also included in the summation [this corresponds to averaging over all initial and final configurations, as was explained in Eq. (2.2)]. For $q=2d$ one has

$$Z = \sum_{l=1}^{q/2} e^{-N\mu_l} \sum_{\{\lambda\}} \prod_{i}^{\text{bonds}} e^{-J_i}(1+z_{i,i+1}\delta_{\lambda_i,\lambda_{i+1}}) \prod_{j}^{\text{sites}} (1+\hat{z}_l\delta_{\lambda_j,\lambda^{(l)}}) \tag{5.16}$$

where now $z_{i,i+1}=e^{J_i}-1$ and $\hat{z}_l=e^{2\mu_l}-1$.

Consider first the case $\mu_l = 0$ for all l and let for a moment all bonds be the same (nonrandom case); then one obtains[30]

$$Z = \sum_{\{\lambda\}} \prod_{i}^{\text{bonds}} e^{-J(N-1)}(1 + z\delta_{\lambda_i, \lambda_{i+1}})$$
$$= e^{-J(N-1)} \sum_{G} z^b q^c \tag{5.17}$$

In the last line the summation goes over all possible graphs such that b is the total number of bonds in a particular graph and c is the number of connected components for this graph (the isolated site is also regarded as a component). Although the expression in the second line in Eq. (5.17) is correct, it is not immediately obvious how it can be actually computed. Here I provide an explicit example of such a calculation, having in mind further generalizations to the case $\mu_l \neq 0$. Following ref. 15, it is convenient to use the identity

$$e^{J\delta_{\lambda,\lambda'}} = \frac{1}{q}\left[(e^J + q - 1) + (e^J - 1)(q\delta_{\lambda,\lambda'} - 1)\right] \tag{5.18}$$

This identity can be also equivalently rewritten as

$$e^{J\delta_{\lambda,\lambda'}} = \frac{\lambda_1}{q}(1 + z\hat{\delta}_{\lambda,\lambda'}) \tag{5.19}$$

where $z = \lambda_1/\lambda_2$ and $\lambda_1 = e^J + q - 1$, $\lambda_2 = e^J - 1$, and $\hat{\delta}_{\lambda,\lambda'} = q\delta_{\lambda,\lambda'} - 1$. Unlike the usual Kronecker delta, the above introduced $\hat{\delta}_{\lambda,\lambda'}$ has a remarkable property which makes it very similar to the summations in the Ising model case. While in Ising case one has $\sum_\sigma \sigma = 0$ and $\sum_\sigma \sigma^2 = 2$, in the present case one has

$$\sum_{\lambda'} \hat{\delta}_{\lambda,\lambda'} = 0 \tag{5.20}$$

$$\sum_{\lambda'} \hat{\delta}_{\lambda,\lambda'}\hat{\delta}_{\lambda',\lambda''} = q\hat{\delta}_{\lambda,\lambda''} \tag{5.21}$$

In, particular, when $\lambda = \lambda''$ one obtains $\hat{\delta}_{\lambda,\lambda} = q - 1$, so that

$$\sum_{\lambda'} \hat{\delta}^2_{\lambda,\lambda'} = q(q-1) \tag{5.22}$$

The properties (5.20) and (5.22) are practically the same as for Ising spins (for $q = 2$ they are in complete agreement with the Ising case) and therefore

significantly simplify the calculations. Using Eqs. (5.19)–(5.22) in (5.16) (with $\mu_l = 0$ for all l), one obtains at once

$$Z = q[1 + (q-1)\, e^{-J}]^{N-1} \tag{5.23}$$

which obviously coincides with Eq. (4.7), as expected. The real power of Eqs. (5.21)–(5.22) is going to be appreciated when $\mu_l \neq 0$. Let, for simplicity, only one of μ_l be left nonzero. This by no means diminishes the generality of the method, as will become apparent shortly. By analogy with Eq. (5.19), one can also write

$$e^{\mu_l \delta_{\lambda, \lambda^{(l)}}} = \frac{\hat{\lambda}_1}{q} [1 + \hat{z}\hat{\delta}_{\lambda, \lambda^{(l)}}] \tag{5.24}$$

where $\hat{\lambda}_1 = e^{\mu_l} + q - 1$ and $\hat{z} = (e^{\mu_l} - 1)/\hat{\lambda}_1$. Now, instead of Eq. (5.18), one gets

$$Z = e^{-\mu_l N} e^{-J(N-1)} \left(\frac{\lambda_1}{q}\right)^{N-1} \left(\frac{\hat{\lambda}_1}{q}\right)^{N} \sum_{\{\lambda\}} \prod_{i}^{\text{bonds}} (1 + z\hat{\delta}_{\lambda_i, \lambda'_{i+1}}) \prod_{j}^{\text{sites}} (1 + \hat{z}\hat{\delta}_{\lambda_j, \lambda^{(l)}}) \tag{5.25}$$

Using the same type of expansion as given by Eqs. (5.7) and (5.8) and employing the Ising-type properties of $\hat{\delta}$'s, one is left with the following result:

$$\frac{1}{q^N} \sum_{\{\lambda\}} \prod_{i}^{\text{bonds}} (1 + z\hat{\delta}_{\lambda_i, \lambda_{i+1}}) \prod_{j}^{\text{sites}} (1 + \hat{z}\hat{\delta}_{\lambda_j, \lambda^{(l)}})$$
$$= 1 + (N-1)\, \hat{z}^2 z + (N-2)\, \hat{z}^2 z^2 + (N-3)\, \hat{z}^2 z^3 + \cdots + \hat{z}^2 z^{N-1} \tag{5.26}$$

which formally coincides with that given in Eq. (5.9), except now z and $\hat{z}$ may have different meanings. Recalling that $e^{\mu_l} + q - 1 \approx q - ip\varepsilon/\sqrt{2} \approx q$ and $e^{\mu_l} - 1 \approx -ip\varepsilon/\sqrt{2}$, we see that in the present case $\hat{z} \approx -ip\varepsilon/q\sqrt{2}$, while

$$z = \frac{1 - e^{-j}}{1 + (q-1)\, e^{-j}} = \frac{1 - \varepsilon m}{1 + (q-1)\, \varepsilon m} \approx 1 - q\varepsilon m$$

Making appropriate rescalings of m and p, one obtains back the Ising case, Eq. (5.9), and thus the rest of the arguments for both the random and non-random cases coincide with that already discussed for the Ising case. This concludes the discussion of Potts-type random copolymers.

6. DISCUSSION

In previous sections a new model of the DSAW was considered in some detail based on earlier reported results of ref. 3. Here, I discuss briefly how this model is related to the models of DSAW[7] already considered in the literature. Let $F(z)$ be the generating function of random walks (Gaussian, self-avoiding, etc.) on some d-dimensional lattice,

$$F(z) = \sum_N C_N z^N \tag{6.1}$$

where C_N is the total number of walks of N steps. Let, furthermore, $G_N(\mathbf{i}, \mathbf{j})$ be the total number of walks starting at the point $\mathbf{i}$ and ending at the point $\mathbf{j}$; then, because of the translational invariance, one obtains $C_N = \sum_i G_N(|\mathbf{i}-\mathbf{j}|) = G_N(\mathbf{p}=0)$, where the last equation represents the $\mathbf{p}=\mathbf{0}$ part of the Fourier-transformed G_N. In the case of the DSAW discussed in ref. 7, one needs to introduce $G_N^R(|\mathbf{i}-\mathbf{j}|) \to G_N^R(\mathbf{p}=0)$, so that, instead of Eq. (6.1), one obtains

$$Z(z, W) = \sum_N \sum_R C_N^R W^R z^N = \sum_N C_N z^N \tag{6.2}$$

where $C_N^R = G_N^R(\mathbf{p}=0)$. Because Eq. (6.2) is equivalent to Eq. (2.10) of ref. 7, the rest of the calculations of ref. 7 follows. Taking into account Eqs. (2.1), (2.2), (3.6), and (4.3), one concludes that

$$\sum_R C_N^R W^R = Z_N(j, 2d) \tag{6.3}$$

where on the rhs of Eq. (6.3), the N-site, one-dimensional, $2d$-component Potts model (in zero field) partition function is written and j is related to W in the way described in the previous sections.

Notice as well that $\langle \mathbf{R}^2 \rangle$, given by Eq. (3.19), is obviously related to $\xi_\perp^2$ given in ref. 7. Unlike the situation in ref. 7, Eq. (3.19) does *not* contain the fugacity z, which, in turn, is determined by the average polymer length $\bar{N}(z, W)$ given by

$$\bar{N}(z, W) = z \frac{\partial}{\partial z} \ln Z(z, W) \tag{6.4}$$

The authors of ref. 7 admit that the explicit form of $\bar{N}$ is rather complicated in general, and they do not provide it. The authors of ref. 7 also notice some nonuniversalities of the scaling functions for DSAW models. They found that these nonuniversalities are caused by the sensitivity of the scaling functions to the choices of lattices.

In Section 5, I have noticed that Eq. (3.19) loses its meaning in the limit $\eta \to 0$. In this limit one obtains $\langle \mathbf{R}^2 \rangle = 0$, which is clearly physically meaningless. On the other hand, when $\eta \to \infty$, the results for $\langle \mathbf{R}^2 \rangle$ become insensitive to the lattice structure. This situation corresponds *exactly* to the scaling limit considered in ref. 7.

Thus, I have demonstrated, that as long as $\eta \neq 0$, the model discussed in the previous sections is well defined and can be used to describe both flexible ($\eta \to 0^+$ but $\eta \neq 0$) and semiflexible ($\eta \to \infty$) polymers.

In addition, use of the results of ref. 3 combined with that of Section 5 allows one to study the nematic order in solutions of random semiflexible polymers. This is going to be presented in a separate publication.

The Ising-like treatment of the Potts model briefly discussed in Section 5 can be extended, in principle, to higher dimensions. In this case, however, the situation is complicated by the fact, that in addition to Eq. (5.22), one is forced to consider quantities such as $\sum_{\lambda'} \hat{\delta}^3_{\lambda,\lambda'}$, etc., which are nonzero, in general. In case of two dimensions it is possible to collect Ising-like graphs, i.e., the set of various closed loops. Now, unlike the Ising case, each loop will be multiplied by a factor of $(q-1)$; see Eq. (5.22). This leads to some nontrivial results to be discussed elsewhere. Formulation of the Ising model in terms of Grassmann-type anticommuting variables is well known.[31] One can perform a similar type of calculation for the Potts model along the lines recently discussed in ref. 32.

Finally, in a previous paper,[3] I have demonstrated a very close connection between the path integral formalisms describing the Dirac particle and the heterotic superstring (which is nothing but the random surface with rigidity, when the Euclidean version of superstrings is considered). The methods I used in ref. 3 are essentially of differential geometry. If such a connection exists between the continuous formulations, one might anticipate that there might be as well some connection between the discrete formulations. At the moment, this is an entirely open problem.

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